Pants Decomposition as Circuit Complexity in 2D (T)QFT

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1. Holographic Complexity
   - Wormhole Growth Paradox
   - Circuit Complexity
   - CA and CV
   - Geometry as a Circuit?
2. TQFT
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3. Complexity in 2D TQFT
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   - Other bordism categories
   - AdS/CFT?
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Holographic Complexity

Wormhole Growth Paradox

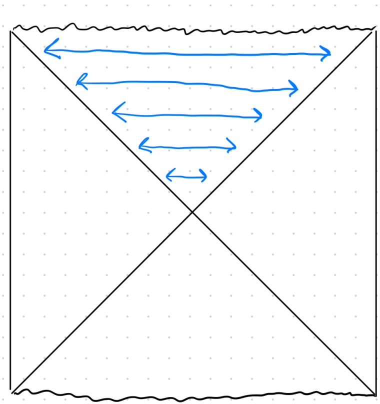

Figure: Growth of the behind the horizon region of a two sided black hole

There are some puzzling aspects to the geometry behind the horizon of a two sided black hole.

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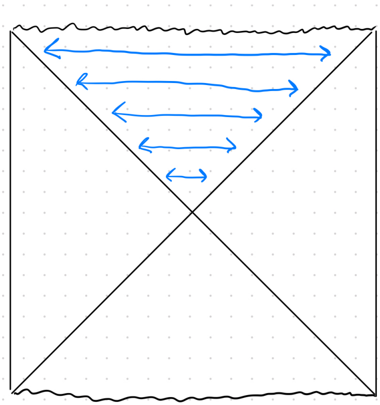

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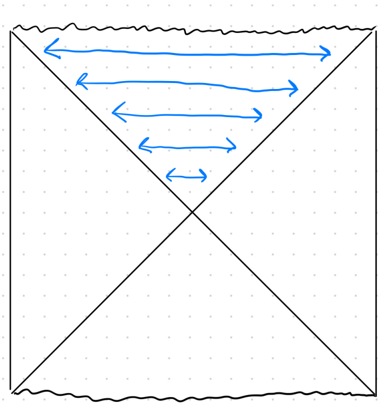

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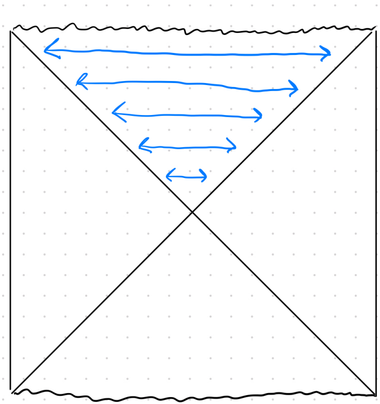

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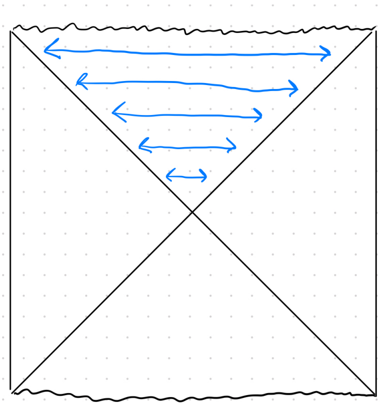

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To resolve these issues, Susskind and collaborators proposed that the geometry behind the horizon is (in some sense) dual to the quantum circuit complexity of the dual state.

Circuit Complexity

What is circuit complexity?

- A gate set is a collection of 'small' unitaries
- A circuit is a formal product of tensor products of gates
- The complexity of a circuit is the number of gates participating in the formal product
- A gate set is universal if any unitary can be approximated by a circuit (to within some tolerance ϵ)
- The complexity of a unitary operator is the smallest complexity among all circuits which approximate it (to within ϵ)
- The complexity of a state $|psi\rangle$, given a reference state $|r\rangle$, is the smallest complexity of any unitary U such that $U|r\rangle = |\psi\rangle$

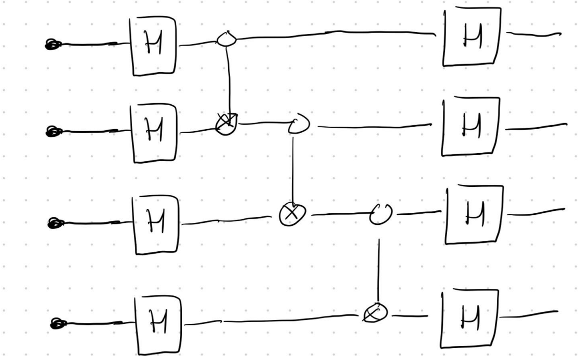

Figure: A circuit with complexity 11

Complexity = Action and Complexity = Volume

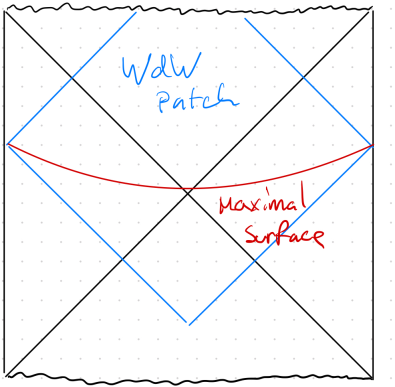

Figure: WDW patch and maximal volume slice of two sided black hole

- complexity = action (CA) proposes that the circit complexity of the CFT state is dual to the action evaluated on the Wheeler-DeWitt (WDW) patch
- complexity = volume (CV) proposes that it is instead dual to volume of a maximal spatial slice
- there have been other proposals, which we will not discuss here

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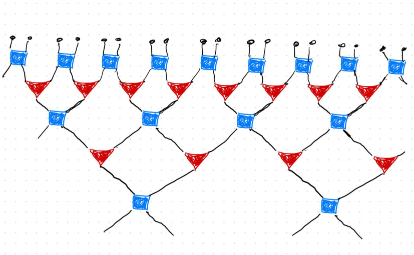

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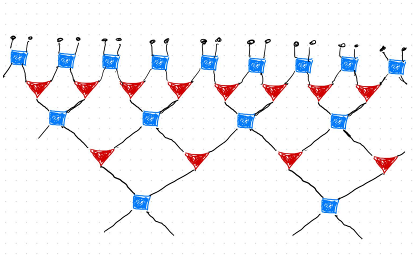

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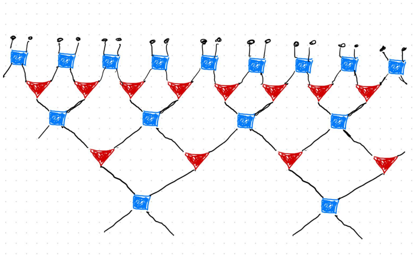

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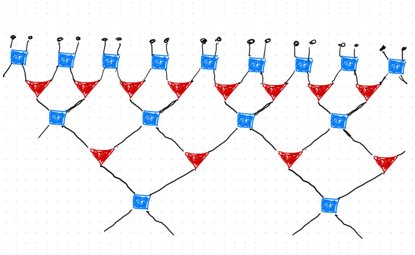

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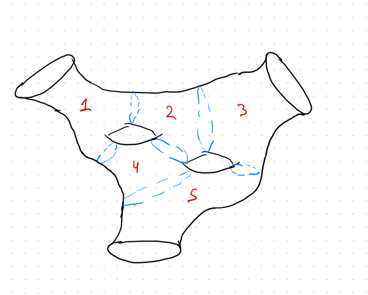

Figure: A surface cut into five pairs of pants

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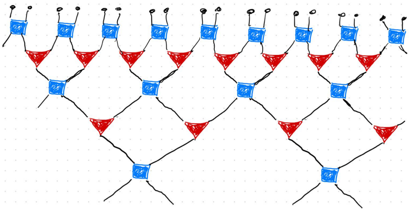

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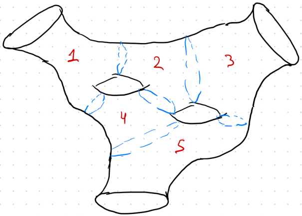

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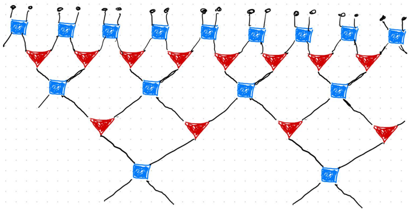

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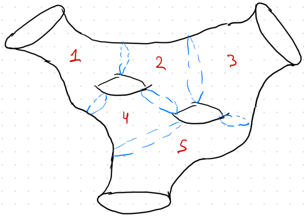

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- However, we will consider another alternative, based on the pants decomposition of a manifold
- (Of course this only works for two dimensions)

TQFT

Category

A category consists of

- A collection of objects (or dots)
- For every pair a, b of objects, a collection of arrows from a to b.
  - We write an arrow $f : a \to b$
- A composition rule $\circ$ which, which takes arrows $f : a \to b$ and $g : b \to c$ and assigns them to a new arrow $g \circ f : a \to c$
- which is associative, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$
- For every object a an identity arrow 1_a such that for all $f : z \to a$ and $g : a \to b$, $1_a \circ f = f$, $g \circ 1_a = g$

We will be interested in symmetric monoidal categories, which additionally have an associative product between objects and arrows.

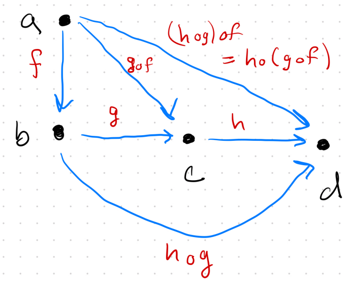


Figure: Some dots and arrows

nCob and Hilb

We will be concerned with two categories in particular

nCob:

- objects are oriented $n-1$ dimensional manifolds
- for manifolds a and b, an arrow $f: a \to b$ is a bordism from a to b
- i.e. an oriented n dimensional manifold with boundary $a \sqcup \bar{b}$, $\bar{b}$ being the orientation reversal of b
- product is disjoint union
- composition is by gluing

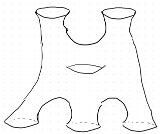


Figure: A 2-bordism from 3 circles to 2

Hilb:

- objects are Hilbert spaces
- arrows are linear operators
- composition is composition of operators as functions
- product is tensor product
- NB the arrows are arbitrary linear operators, they need not be e.g. unitary

Functor

A functor F from category A to category B

- assigns to each object a in A an object $F(a)$ in B
- assigns to each arrow $f : a \to b$ in A an arrow $F(f) : F(a) \to F(b)$ in B
- respects composition between arrows
- a symmetric monoidal functor (between symmetric monoidal categories) additionally repsects products

One may define an n dimenional topological quantum field theory (TQFT) as a symmetric monoidal functor between nCob and Hilb. For bordisms this functor essentially just computes the path integral over the bordism.

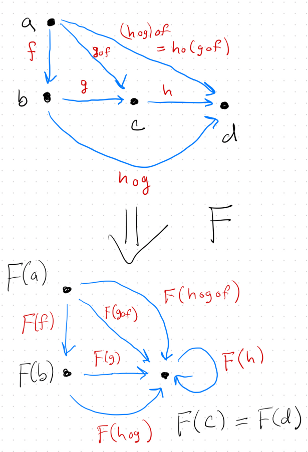


Figure: A functor

Complexity in 2D TQFT

Complexity in 2-Cob

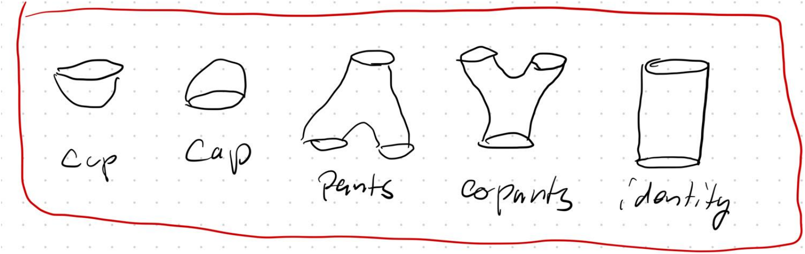
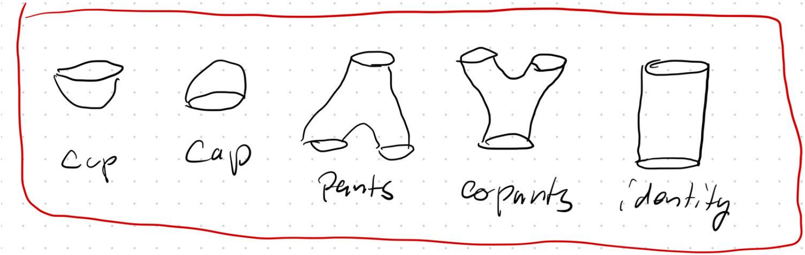


Figure: Gates in 2Cob

Complexity in 2-Cob

- Every bordism can be decomposed (by cutting) into a composition of 5 fundamental pieces which we will call gates



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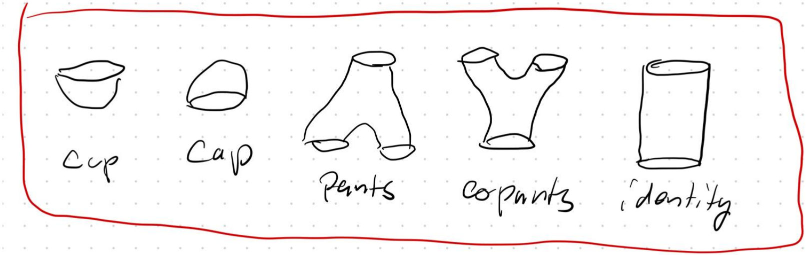

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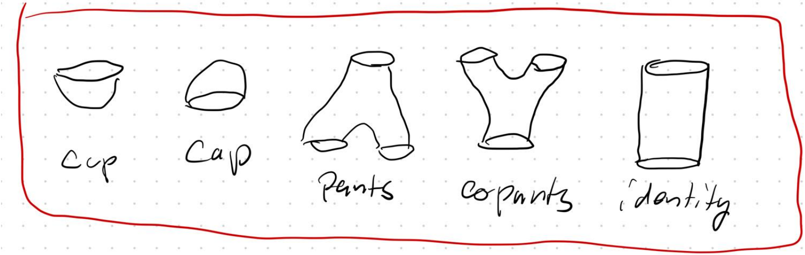
- With a few exceptions (such as the sphere or torus), a bordism with n boundaries and genus g will then have complexity $-\chi(n,g) = 2g - 2 + n$.



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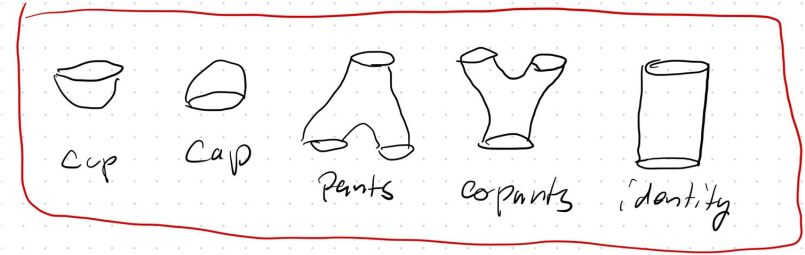
- In these (non-exceptional) cases, this complexity is just the number of pants in a pants decomposition.



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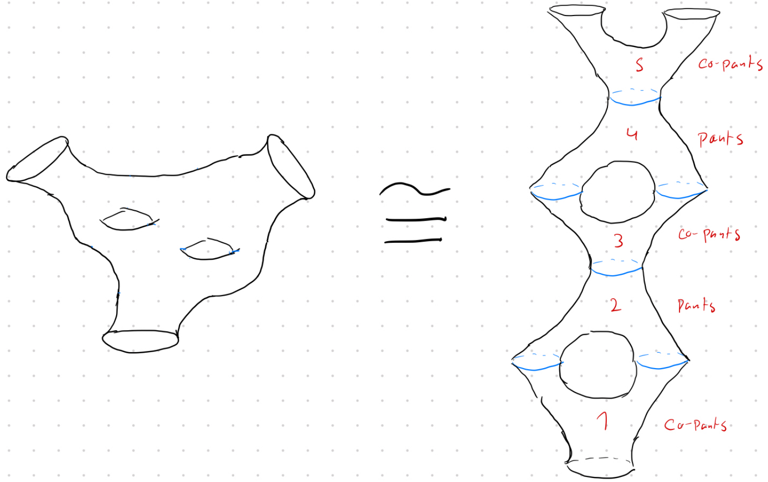

Figure: A bordism of complexity 5

Circuit Complexity in Hilb from 2Cob

Can we use this gate counting in 2Cob to define a complexity in Hilb?

- A TQFT will assign each of the gates we discussed above to a linear operator
- However, these gates will not be unitary (they are maps between Hilbert spaces of different dimensions)
- In addition, they will not be universal: most linear operators do not have any corresponding bordism.
- It is also not universal with respect to states, i.e. these gates are not enough to contruct an arbitrary state from some reference state
- In the preprint, we study the degree to which these gates fail to be universal for different sets of TQFTs
- The states and operators that can be constructed are essentially those which can be constructed from path integrals

Future Directions

Other bordism categories

One might consider other categories of bordisms, including with more structure, corresponding to richer QFTs.

Examples:

- higher dimensional bordisms $\leftrightarrow$ higher dimensional TQFT
- bordisms with lower dimensional defects $\leftrightarrow$ defect TQFTs
- bordisms with spin structure $\leftrightarrow$ spin TQFTs
- bordisms with conformal structure $\leftrightarrow$ conformal field theories

CFTs are of course of particular interest given AdS/CFT

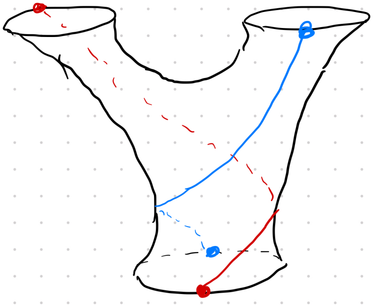

Figure: A 2 dimensional bordism with 1 dimensional defects

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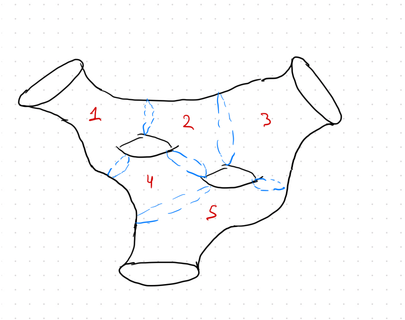

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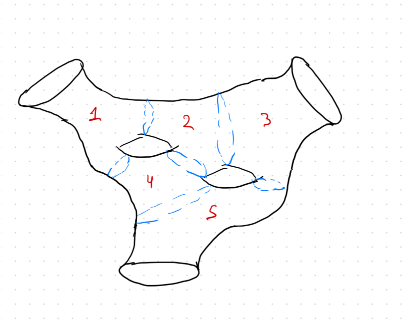

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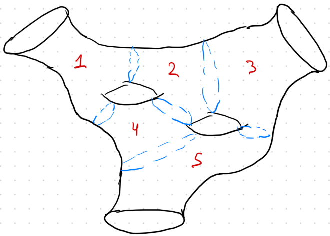

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- Does this even make any sense?

Chern-Simons/WZW?

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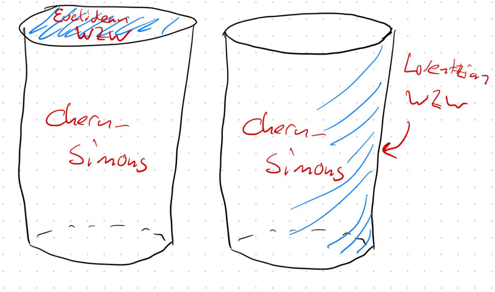

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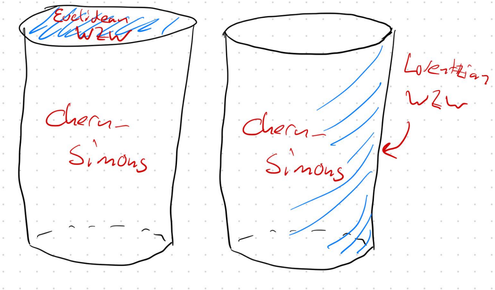

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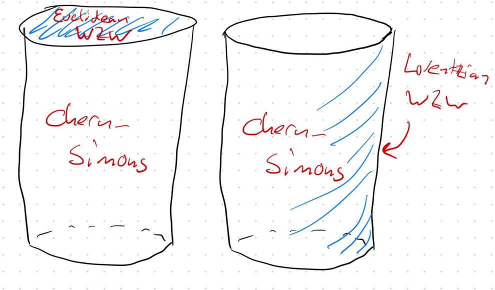

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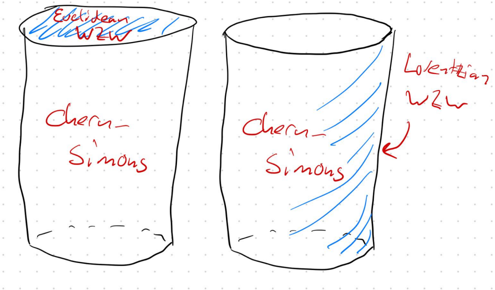

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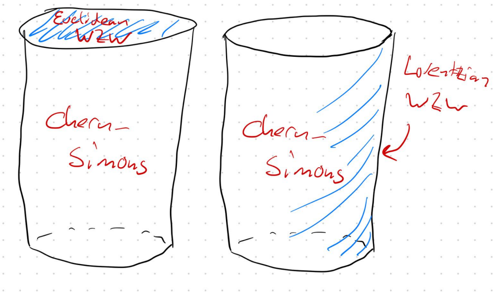
- As such, it seems that one could perhaps identify the complexity of the CS state with the bordism complexity of a spatial slice
- NB that because WZW is not a TQFT, the it will need a richer set of gates to take care of the extra (conformal) structure



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