Holographic Complexity

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24 April 2018

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   - Holographic Complexity
2. Holographic Complexity and Non-Commutative Field Theory
   - Introduction
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   - Effect of Non-Locality on Complexity
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Introduction

The AdS/CFT Correspondence

In 1999, Juan Maldacena argued based on string theory that $\mathcal{N} = 4$ super Yang-Mills (SYM) in $3 + 1$ dimensions with gauge group SU(N) is dual to type IIB string theory on $AdS_5 \times S^5$. This became the first example of so-called AdS/CFT duality between a gravity theory in an asymptotically anti de-Sitter (AdS) space time (times a compact manifold) and a conformal field theory (CFT) living 'on the conformal boundary' of that spacetime.

- These theories are dual in that the partition functions are 'the same'.
- In a certain limit (usually taken), the rank N of the gauge group becomes large, and the string theory approximately reduces to classical Einstein gravity.
- At low energies we can get rid of compact manifold by a Kaluza-Klein reduction.
- When the field theory is strongly coupled, gravity theory is weakly coupled.
- So in the end we have a duality between a strongly coupled field theory in d dimensions and classical gravity (plus KK modes) in asymptotic AdS_{d+1}.
- UV of the field theory becomes IR of the gravity theory.
- A particular classical geometry is actually dual to a particular (possibly time dependent) state on the boundary.

Applications of the AdS/CFT Correspondence

One important application of AdS/CFT has been to study strongly coupled field theories.

- Boundary theory is an SU(N) gauge theory, so there have been attempts to use AdS/CFT to study QCD at strong coupling.
- There is a hope to be able to understand strongly coupled condensed matter systems.
- In particular, it is easy to study the entanglement structure of the boundary state in AdS/CFT

On the other hand, there is an effort to use AdS/CFT to say something about quantum gravity.

- AdS/CFT would seem to imply black holes are governed by unitary dynamics, contrary to arguments by e.g. Hawking.
- There has been work deriving the einstein equations as a consequence of the first law of entanglement entropy on the boundary.
- Perhaps geometry emerges from the entanglement structure of quantum UV degrees of freedom?

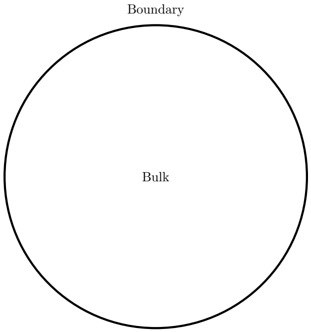
Thermodynamics in AdS/CFT

Under the AdS/CFT dictionary, a black hole in the bulk is dual to a thermal state on the boundary, with the temperature, entropy, etc. of the boundary state identical to those of the black hole.

- Consider an asymptotically AdS black hole geometry: $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$
- Wick rotating to euclidean time t_E, we see that in order to avoid a conical singularity at the horizon, we must identify $t_E \sim t_E + \beta$, with β the inverse black hole temperature.
- Because the bulk and boundary partition functions are the same, we have in a saddle point approximation $\ln(Z) \approx -S_{\text{cl}}$, where S_{cl} is the Euclidean Einstein-Hilbert action evaluated on our particular solution.
- Evaluating $U = -\frac{\partial \ln Z}{\partial \beta}$, we find that $U := \langle H \rangle = M$, where H is the boundary Hamiltonian, and and M is the mass of the black hole.
- Similarly, one may find the entropy by $S = -\frac{\partial (T \ln Z)}{\partial T}$ to discover that the entropy of the boundary state is simply the entropy of the black hole.
- We likewise have that the charge of the black hole is dual to the boundary chemical potential.

Entanglement and Bulk Reconstruction

I mentioned before that AdS/CFT can also give the entanglement structure of the boundary state. This is given by the so called Ryu-Takayanagi prescription:



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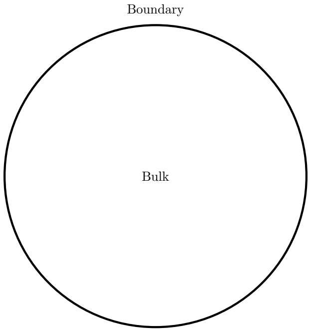

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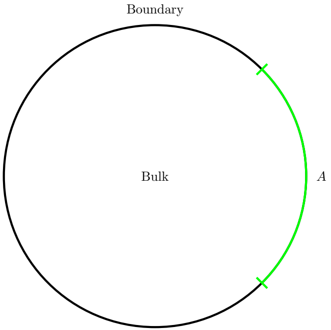

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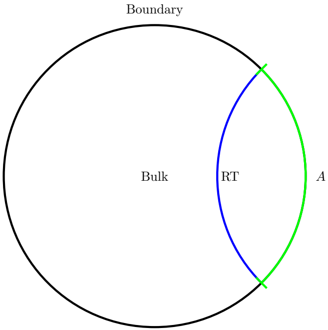

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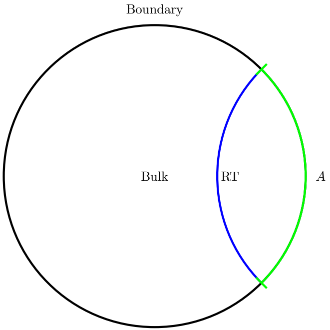

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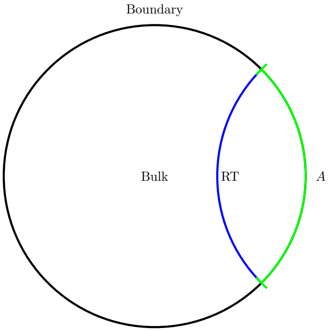

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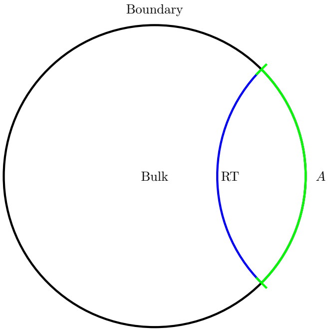

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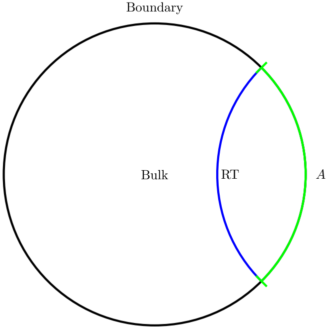

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By considering all possible subregions and their respective entanglement entropies, we may hope to reconstruct the dual bulk geometry from field theory data.

Behind the BH Horizon

We state before that a black hole geometry is dual to a thermal state. Strictly speaking, it is geometry outside the horizon which is dual to a thermal state.

- The (geodesically) complete geometry is a two sided black hole geometry
- This geometry has two boundaries, with a CFT on each boundary. The Hilbert space is given by $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- The state dual to this configuration is a purification of the thermal state known as the thermofield double state:

$$\psi = \sum_n e^{-\beta E_n/2} \left|E_n\right\rangle_L \otimes \left|E_n\right\rangle_R$$

- Outside of horizon geometry corresponds in some sense to the entanglement structure of the thermal state.
- What about the the geometry behind the horizon?

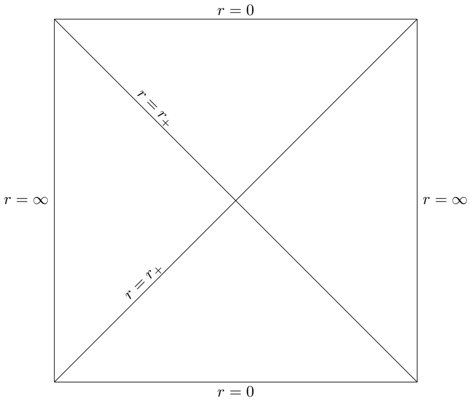


Figure : Two sided black hole

Holographic Complexity

There are some puzzling aspects to the behind the horizon geometry

- No region defined entirely on the left or right side is sensitive to this part of the geometry
- The volume behind the horizon on a spatial slice continues growing long past the thermal time.

To resolve these issues, Leonard Susskind proposed that the geometry behind the horizon is dual to the *quantum circuit complexity* of the dual state.

- Specifically, Susskind proposed that the complexity of boundary state is dual to the volume of a maximal spatial slice of the bulk. (Complexity = volume)
- Like the volume, the complexity of the state obtained from the TFD state by unitary time evolution is expected to incrase long past the thermalization time
- In fact it should grow for a time exponential in the number of degrees of freedom
- Most of the increase in such a slice with time occurs behind the horizon.

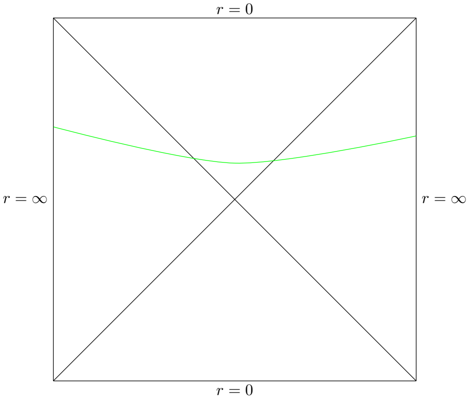


Figure : Maximal spatial slice

Quantum Circuit Complexity

What is circuit complexity?

- Consider a Hilbert space $\mathcal{H}$, e.g., the Hilbert space for N quantum bits.
- A universal gate set $\{g_i\}$ for $\mathcal{H}$ is a set of unitary operators on the Hilbert space such that any unitary U acting on $\mathcal{H}$ can be approximated by some product $\prod_i g_{\alpha_i}$ to within a small tolerance ϵ.
- Such a product of gates is referred to as a quantum circuit.
- The quantum circuit complexity of a unitary U is then the minimum number of gates needed to approximate U to within the tolerance.
- In the example of qubits, one typically considers gates that act on a single qubit or pairs of qubits at a time.
- Given some reference state $|\psi_R\rangle$, one may define the complexity of a state $|\psi\rangle$ as the minimum of complexity $C(U)$ over all unitaries U such that $|\psi\rangle = U\,|\psi_R\rangle$.

The Switchback (Butterfly) effect

- Given the time evolution operator $U(t)$ and some other operator W, we may construct a *precursor operator* $U(t)WU(-t)$.
- The expectation value of this operator is the overlap between the current state, and the state we would have obtained at the present had the operator W been inserted a time t in the past.
- The complexity of this operator is lower bounded by the sum of the complexities of the individual operators: $\mathcal{C}(U(t)WU(-t)) \leq 2\mathcal{C}(U) + \mathcal{C}(W)$.
- If W is Haar random, this bound is saturated with overwhelming probabillity.
- On the otherhand, clearly if W is the identity the complexity is 0.
- If W is a local operator, then
  - for small t, the complexity is very small (nearly total cancelation).
  - As t increases, the local operator W gets more and more scrambled over the whole Hilbert space.
  - At large t (much larger than the scrambling time), the complexity of the precursor operator grows at the same rate as ordinary time evolution.
- The insertion of an operator W in the boundary theory in AdS/CFT is dual to a shockwave in the bulk originating at the boundary
- Shockwave geometries may thus be used to test whether complexity = volume reproduces the switchback effect.
- Complexity = volume has in fact been found to be consistent with the switchback effect in a wide variety of circumstances.

Complexity $=$ Action

Complexity $=$ volume has a few unpleasant features

- For example, in order to reproduce the correct boundary behavior, the volume must be multiplied by a non-universal length scale

One might seek an alternative proposal, which still captures something about the behind the horizon geometry, but which does not have these features. In fact, such an alternative has been proposed by Susskind et al., and it goes by 'complexity $=$ action.'

- According to complexity $=$ action, the complexity is dual to the action of a so-called 'Wheeler-DeWitt' (WDW) patch.
- Because this is an action, it can be nondimensionalized with some multiple of $\hbar$.
- A universal choice for this coefficient is consistent with the expected large temperature behavior.
- The action of a WDW patch also behaves in the appropriate way in the presence of shockwaves, so it still reproduces the switchback effect.

The WDW patch

- The WDW patch is defined by a spatial slice of the boundary.

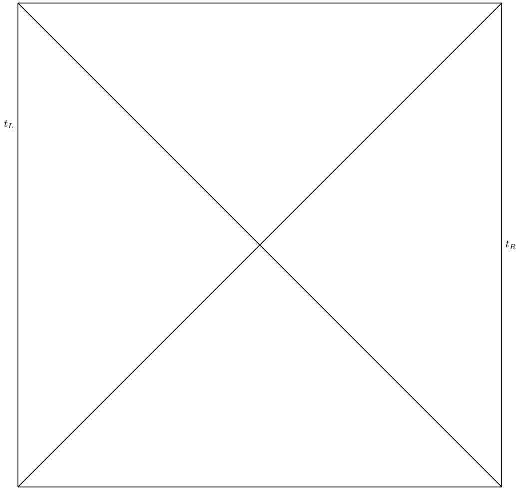
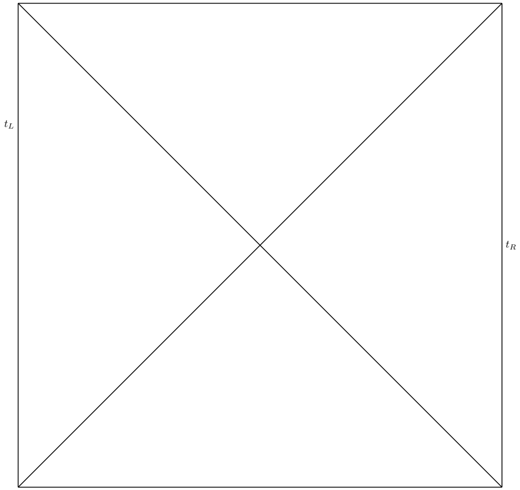


Figure : Boundary times t_L and t_R

The WDW patch

- The WDW patch is defined by a spatial slice of the boundary.
- For a two-sided black hole, this can be given by a left time t_L and a right time t_R.



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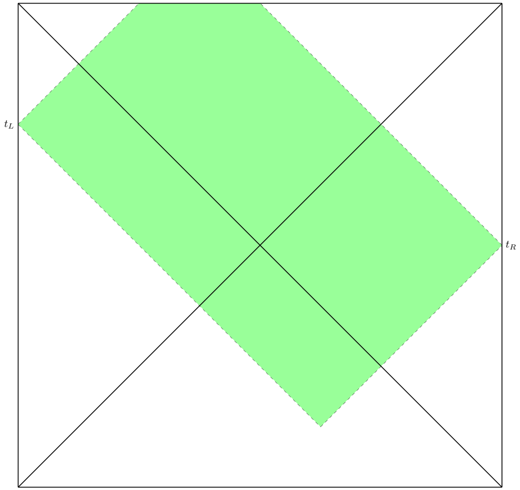

Figure : The WDW patch defined by boundry times t_L and t_R

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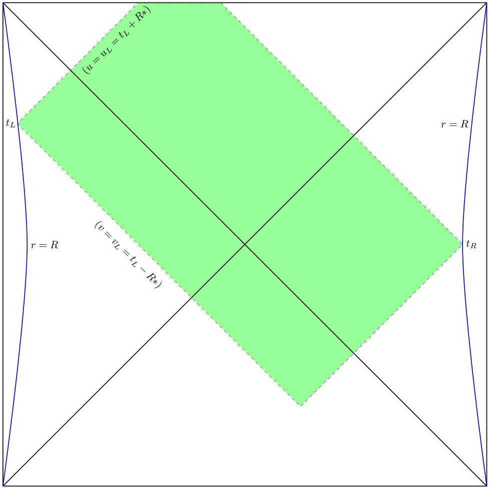

Figure : The WDW patch defined by boundry times t_L and t_R and a cutoff R

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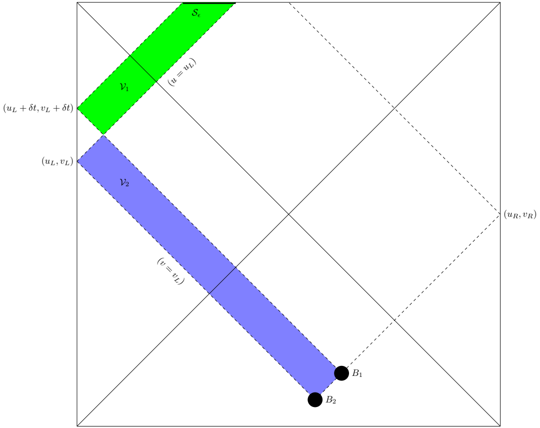

Figure : Two WDW patches separated by δt. In thi figure, we have suppressed the cutoff

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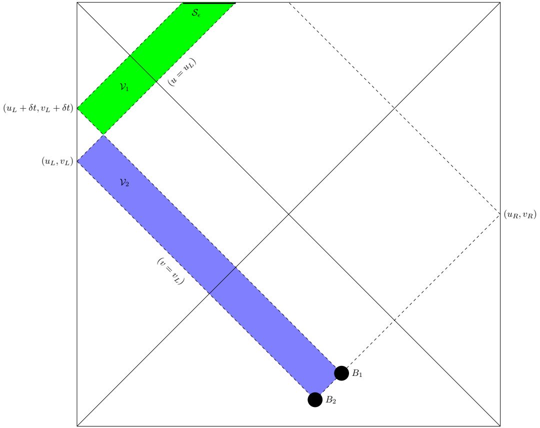
- We will only be interested in the rate of change of the action, as t_L increases.
- This may be computed by subtracting the actions of two WDW patches, whose left time is separated by δt, and then taking the limit where $\delta t \to 0$.



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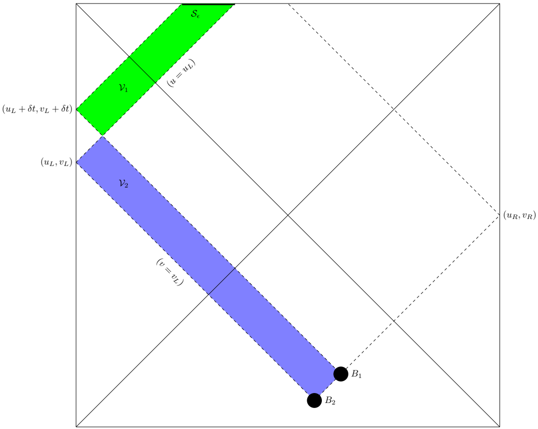
- Often we will be interested in the rate of change of the action, as t_L increases.
- This may be computed by subtracting the actions of two WDW patches, whose left time is separated by δt, and then taking the limit where $\delta t \to 0$.
- This difference of actions decomposes to the difference of two bulk pieces, a piece from the spacelike boundary of a near singularity cutoff, and two codimension two contributions from the past corners of the patches.



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Holographic Complexity and Non-Commutative Field Theory

Introduction

It would be nice to have a first principles derivation of the holographic dual of circuit complexity. Unfortunately, this has not yet been done and is probably hard to do. In the absence of such a derivation, we would like to have some novel contexts where we can 'test' the holographic complexity by comparing to expected behaviour.

- One possibility we will discuss here is non-commutative field theory, i.e. field theory living on a non-commutative manifold
- In particular, we will be interested in Non-commutative ($\mathcal{N} = 4$) super Yang-Mills (NCSYM).
- A gravity solution dual to NCSYM was derived in by a number of authors by Hashimoto and Itzhaki (1999) and was later discussed by Maldacena and Russo (1999).
    - Themodynamics (Cai & Ohta (2000), Fischler et al. (2000))
    - Fast Scambling (Edalati et al. (2012))
    - Entanglement entropy (Fischler et al. (2013), Karczmarek et al. (2013))
- Also, generalizations of this system to other numbers of dimensions have also been considered in, e.g., Alishahiha et al. (1999) and Berman et al. (1999).

Non-Commutative Geometry

What is a non-commutative manifold?

- A manifold in which the coordinates are taken to be non-commuting, i.e. given coordinates x_i and x_j, we introduce a non-trivial commutator $[x_i, x_j]$ between them.
- Ordinary quantum mechanics is a theory on a non-commutative manifold, where the manifold $\mathcal{M}$ is the symplectic manifold of a Hamiltonian system and $[x_i, x_j] = i\hbar\, \omega^{ij} x_i x_j$, where ω is the symplectic form on $\mathcal{M}$.
- One may do quantum mechanics on a non-commutative spatial manifold by introducing a non-zero commutator between distinct directions, e.g. in 2 dimensions we can have $[x, y] = ia^2$. In this case a is called the *Moyal scale*
- This new commutator implies a new uncertainty relation $\Delta x \Delta y \geq ia^2$, and so particles cannot be localized at spatial scales larger smaller than a. This is one way of seeing that non-commutative theories are non-local.
- One may define fields on a non-commutative manifold by replacing ordinary products of fields $\phi(x)\psi(y)$ by the so-called Moyal product: $(\phi * \psi)(x) = \exp(ia^2 \nabla_x \nabla_{x'})\phi(x)\psi(x')\Big|_{x'=x}$
- The theory of charges in a conducting sheet with a normal magnetic field (A quantum Hall system) becomes non-commutative when restricted to the lowest Landau level.

The gravity dual to NCSYM

The gravity dual to NCSYM is obtained in a manner similar to that of standard $\mathcal{N} = 4$ SYM.

- Consider a stack of D3-branes
- We turn on a 2-form B parallel to the two spatial dimensions of the branes
- This can be achieved by applying a combination of T-dualities and Gauge transformations
- The B-field has the effect of introducing non-commutativity in the worldbrane theory
- The near horizon geometry in the usual limit gives the gravity dual, which is a IIB SUGRA solution.

The resulting solution, at finite teperature, is given in Einstein frame as follows:

$$ds^2 = \alpha' \left[\left(\frac{r}{L}\right)^2 \left(-f(r)dt^2 + dx_1^2 + h(r)(dx_2^2 + dx_3^2)\right) + \left(\frac{L}{r}\right)^2 \left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right)\right], \tag{1}$$

$$e^{2\Phi} = \hat{g}_s^2 h(r), \quad B_{23} = B_\infty(1 - h(r)), \quad C_{01} = -\frac{\alpha' a^2 r^4}{\hat{g}_s R^2}, \quad F_{0123r} = \frac{4\alpha'^2 r^3}{\hat{g}_s R^4} h(r) \tag{2}$$

$$f(r) = 1 - \left(\frac{r_+}{r}\right)^4, \quad h(r) = \frac{1}{1 + a^4 r^4}, \quad B_\infty = -\frac{\alpha'}{a^2 L^2}. \tag{3}$$

Here L is the AdS length scale, r_+ is the bulk coordinate of the horizon, $\hat{g}_s$ is the and closed string coupling, and a is the Moyal scale (i.e. $[x_2, x_3] = ia^2$ on the boundary).

The gravity dual to NCSYM

A few additional notes about the gravity dual:

- The bulk coordinate r has units of inverse length.
- Though the boundary field theory lives on a non-commutative manifold, the bulk geometry is commutative.
- The metric degenerates at the boundary. This should be fine, however, provided we always work with a finite cutoff.
- The dimension of the Hilbert space in the dual theory was found to be independent of the Moyal scale by Maldacena and Russo (1999).

We will now consider an intuition based heuristic argument that we should expect the complexity at a given (late) time should be higher in NCSYM than in it's commutative counterpart.

Non-Commutativity and Complexity: A heuristic argument

- Consider the unitary operator U which translates our state in time by a small time δt.

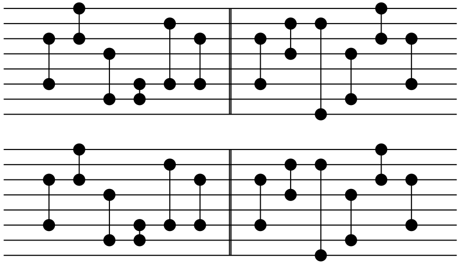

Figure : A small piece of a circuit implementing time translations on a local (top) and non-local (bottom) theory. This cartoon is inspired by another which appears in certain talks by Adam Brown.

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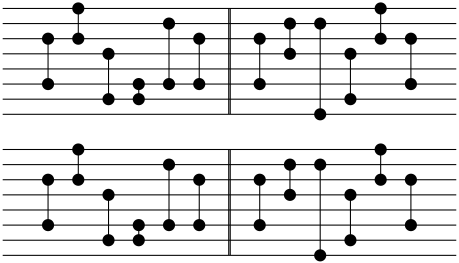

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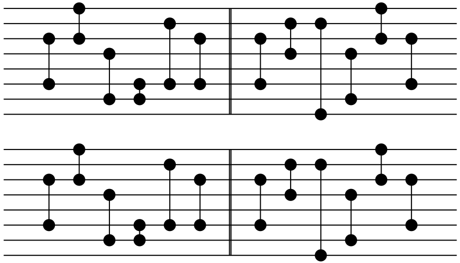

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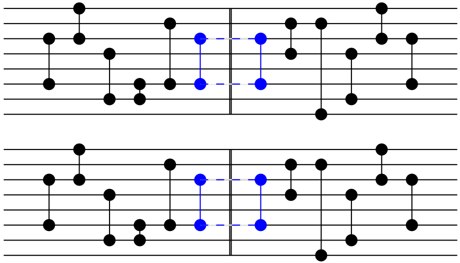
- This circuit is generally non-optimal however, as there will be gates which cancel between the beginning and end of successive copies of Q.



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- At late time t, the circuit Q^N approximates U^N, where $N = t/\delta t$.
- This circuit is generally non-optimal however, as there will be gates which cancel between the beginning and end of successive copies of Q.
- These cancelations lead to a circuit for the same operator of lower complexity

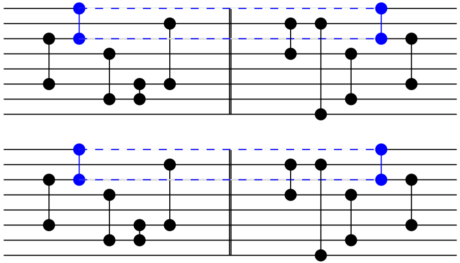


Figure : A small piece of a circuit implementing time translations on a local (top) and non-local (bottom) theory. This cartoon is inspired by another which appears in certain talks by Adam Brown.

Non-Commutativity and Complexity: A heuristic argument

- Consider the unitary operator U which translates our state in time by a small time δt.
- Consider further an optimal circuit Q implementing U.
- At late time t, the circuit Q^N approximates U^N, where $N = t/\delta t$.
- This circuit is generally non-optimal however, as there will be gates which cancel between the beginning and end of successive copies of Q.
- These cancelations lead to a circuit for the same operator of lower complexity
- In a non-local theory (such as a non-commutative theory), fewer operators commute past one another, and so there will be more obstruction to such cancelations, leading to a higher final complexity.

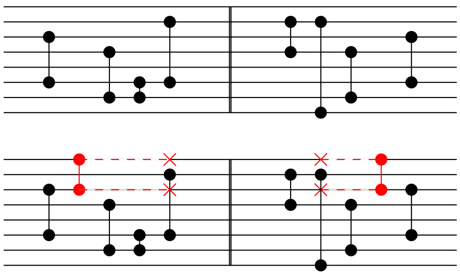


Figure : A small piece of a circuit implementing time translations on a local (top) and non-local (bottom) theory. This cartoon is inspired by another which appears in certain talks by Adam Brown.

Results

Now that we have this naïve expectation, we will compare with the complexity in NCSYM according to CA. We will find it convenient to define the parameters $b = ar_+ = \pi a T$, $\rho = \frac{r_B}{r_+}$, and $\gamma = \frac{c\bar{c}\sqrt{\hat{g}_s}L^2}{\alpha' \pi^2 T^2}$.

- $T = \pi r_+$ is the temperature
- c and $\bar{c}$ are the normalizations of the null generators

With these conventions, the complexification rate after the critical time is given by

$$\dot{C} = \frac{\Omega_5 V_3 r_+^4}{(2\pi)^7 \hat{g}_s^2} \left(\frac{-2\log(1 + b^4 \rho^4)}{b^4} + 4\rho^4 + 6 + 3(1 - \rho^4) \log \left| \frac{\gamma \rho^2}{(1 + b^4 \rho^4)^{1/4}(1 - \rho^4)} \right| \right). \tag{4}$$

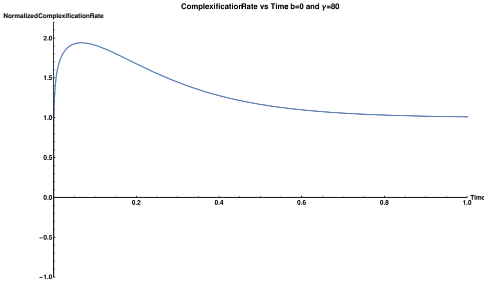
Here V_3 is the volume of a spatial slice of the boundary theory (which is infinite) and Ω_3 is the volume of a unit 5-sphere. In what follows, we will normalize the rates by the $b = 0$ result.

D3-brane results: Finite time behavior

In the commutative limit, we see similar behavior to that reported by Carmi et al. (2017).

- Logarithmic divergence at the critical time.
- Reaches true global maximum in order one time (in thermal units).
- approaches asymptotic value from above.

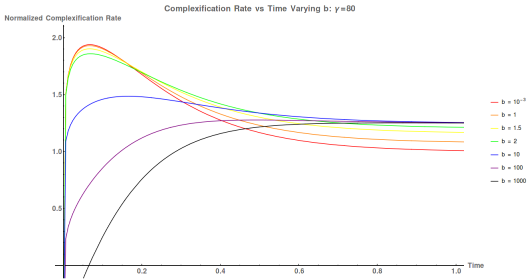
This behavior persists for all values of the Moyal scale but as the Moyal scale becomes very large, the true maximum shrinks and the asymptotic value increases, so that the gap between them decreases

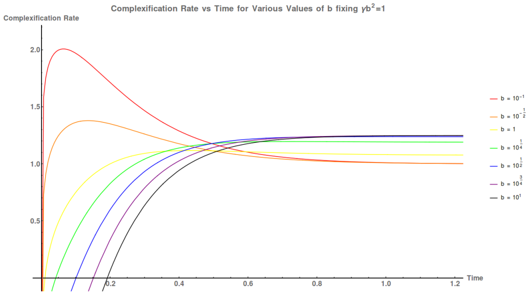


This behavior calls into question the normalization to complexity = action as set by Brown et al. (2015), though the logic that to that normalization would seem already to be contradicted by Cottrell and Montero (2017).

Finite time behavior

Here we see how the finite time behavior changes as we vary the parameters.





D3-brane results: Late time limit

We take the late time limit by sending r_b to r_+ (i.e. we send ρ to 1). In this limit, the normalized complexification rate becomes

$$\dot{C}_{\text{normalized}}\Big|_{t\to\infty} = \frac{5}{4} - \frac{\log(1+b^4)}{4b^4}. \tag{5}$$

Recalling that $b = \pi a T$, we see that at a fixed temperature, as we send the Moyal scale to infinity, we get $5/4$, which given the normalization scheme tells us that we get exactly a 25% enhancement in of the complexification rate in the large Moyal scale regime.

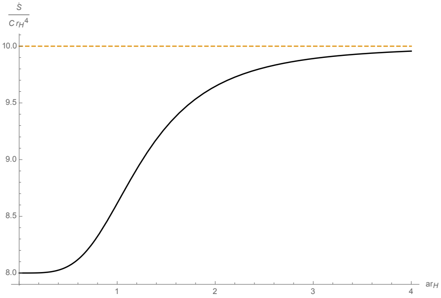


Figure : Late time action growth rate normalized by $C = \frac{\alpha^4 \Omega_5 V_3}{\hat{g}_s^2}$ and extra r_H dependence, versus ar_+, which is the Moyal scale measured in units of thermal length.

Results for other values of p

As discussed before, the geometry we have considered so far can be generalized by starting with stacks of $Dp - branes$ with for other values of p. For $p \neq 3$, the resulting geometry is not asymptotically AdS, even in the commutative limit. We considered $2 \leq p \leq 5$. For $p \geq 4$ there is the possibility of introducing non-commutativity between multiple pairs of coordinates. The table below summarizes our results for the late time rate of change of the complexity density, with a common normalization for all results. Here m indicates the number of pairs of non-commuting coordinates on the boundary.

p	$m = 0$	$m = 1$	$m = 2$
2	12	12	-
3	8	10	-
4	5	5	8
5	4	5	6

Conclusions

- For $p = 3, 5$ we do see an increase at late time, as expected.
- Though we did not see an increase for $p = 2$ or for $p = 4$ with a single non-trivial commutator, at least we did not see a decrease either.
- Overall, the results are consistent with the heuristic argument above.
- This result is in tension with the idea that commutative black holes are the fastest possible computers
- In future work, we plan to repeat our calculations for complexity = volume.

Further/Future work

Further Work

A few other things we are thinking about in Holographic Complexity:

- Geometric properties of maximal volumes:
  - A 'second law' for complexity behind a future horizon.
  - Positivity of mixed time derivative $\rightarrow$ monotonicity of complexification rate.
  - super-additivity
- Mixed state complexity from field theory.
- Complexity and holographic complexity in other non-local field theories